Wave function statistics in open chaotic billiards

Piet W. Brouwer
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853-2501, USA
(Received 24 February 2003; published 21 October 2003)

We study the statistical properties of wave functions in a chaotic billiard that is opened up to the outside world. Upon increasing the openings, the billiard wave functions cross over from real to complex. Each wave function is characterized by a phase rigidity, which is itself a fluctuating quantity. We calculate the probability distribution of the phase rigidity and discuss how phase rigidity fluctuations cause long-range correlations of intensity and current density. We also find that phase rigidities for wave functions with different incoming wave boundary conditions are statistically correlated.

DOI: 10.1103/PhysRevE.68.046205 PACS number(s): 05.45.Mt, 05.60.Gg, 73.23.Ad

I. INTRODUCTION

Microwave cavities have been used as a quantitative experimental testing ground for theories of quantum chaos [1]. In quasi-two-dimensional cavities, the component of the electric field perpendicular to the surface of the cavity satisfies a scalar Helmholtz equation that is formally equivalent to the Schrödinger equation. Since the Helmholtz equation is real, the microwave electric field in the cavity is real as well. A real field serves as a model for an electronic wave function in the presence of time-reversal symmetry and spin-rotation invariance. Complex field patterns, which model the wave function of an electron in a magnetic field, can be obtained making judicious use of magneto-optical effects [2,3]. Alternatively, complex "wave functions" can be observed as traveling waves in open microwave cavities [4–6]. Measured distributions of real and complex wave functions in microwave cavities with chaotic ray dynamics, where, traditionally, "complex" means that the time-reversal symmetry is fully broken and the phase of the wave function has no long-range correlations, agree with a theoretical description in terms of a random superposition of plane waves [7], as well as with random matrix theory [8] and the supersymmetric field theories [9].

Recently, it has become possible to study the full crossover from real wave functions to complex wave functions using microwave techniques [5,6,10]. The crossover regime is qualitatively different from the "pure" cases of real or fully complex wave functions. Unlike in the pure cases, the statistical distribution of wave functions in the crossover regime depends on the way the statistical ensemble of wave-function elements is obtained: whether variations are taken with respect to the coordinate **r**, the frequency ω, or both. Whereas the theoretical work has been roughly equally divided between the two approaches (explicitly or implicitly), experiments usually need the additional average over frequency to obtain sufficient statistics [2,3,10,11] (see, however, Refs. [5,6] for an exception).

In general, a complex wave function may be written as

$$\psi(\mathbf{r})=e^{i\phi}[\psi_{\mathbf{r}}(\mathbf{r})+i\psi_{\mathbf{i}}(\mathbf{r})], \tag{1}$$

where $\psi_{\mathbf{r}}$ and $\psi_{\mathbf{i}}$ are orthogonal but need not have the same normalization [12]. The ratio of $\psi_{\mathbf{r}}$ and $\psi_{\mathbf{i}}$ is parametrized in terms of the normalized scalar product of ψ and its time reversed,

$$\rho=\frac{\int d\mathbf{r}\psi(\mathbf{r})^2}{\int d\mathbf{r}|\psi(\mathbf{r})|^2}=e^{2i\phi}\frac{\int d\mathbf{r}|\psi_{\mathbf{r}}(\mathbf{r})|^2-|\psi_{\mathbf{i}}(\mathbf{r})|^2}{\int d\mathbf{r}|\psi_{\mathbf{r}}(\mathbf{r})|^2+|\psi_{\mathbf{i}}(\mathbf{r})|^2}. \tag{2}$$

The square modulus $|\rho|^2$ is known as the "phase rigidity" of the wave function ψ [13]. Real wave functions have $\rho=1$, whereas $\rho=0$ if ψ is fully complex, i.e., $\psi_{\mathbf{r}}$ and $\psi_{\mathbf{i}}$ have the same magnitude. If the average is taken over the coordinate **r** only, whereas the frequency ω of the wave function is kept fixed, the wave-function distribution follows by describing $\psi_{\mathbf{r}}$ and $\psi_{\mathbf{i}}$ as random superpositions of standing waves [4,14,15]. The resulting wave-function distribution depends parametrically on the phase rigidity $|\rho|^2$. Using a microwave billiard with a movable antenna, Barth and Stöckmann have measured such a "single-wave-function distribution" and found good agreement with the theory, obtaining ρ from an independent measurement [5]. It is the fact that ρ is different for each wave function that leads to the different results for averages over **r** only and over both **r** and ω, an average over frequency involves an additional average over ρ. Such a full wave-function distribution, which needs theoretical input beyond the ansatz that each wave function ψ is a random superposition of plane waves, was first calculated by Sommers and Iida for the Pandey-Mehta Hamiltonian from random-matrix theory [16] and by Fal'ko and Efetov [17,18] for a disordered quantum dot in a uniform magnetic field.

In addition to being responsible for the difference between probability distributions obtained from an average over position or from an average over position and frequency, fluctuations of the phase rigidity $|\rho|^2$ have been identified as the root cause for several striking phenomena in the crossover regime, such as long-range intensity correlations [18] and a non-Gaussian distribution of level velocities [13]. Further, the existence of correlations between phase rigidities of different wave functions causes long-range correlations between wave functions at different frequencies [19].

Here, we consider the real-to-complex crossover for wave functions in a billiard that is opened up to the outside world, and calculate the probability distribution of phase rigidities for this case. Although time-reversal symmetry is not broken on the level of the wave equation itself, it is broken by the fact that one looks at a scattering state with incoming flux in one waveguide only [4]. As we show here, random wave functions in open cavities also have a fluctuating phase rigidity, and, hence, exhibit the same variety of phenomena as those in cavities with broken time-reversal symmetry, while they are much easier to generate in microwave experiments [5,6]. An additional advantage of the open-billiard geometry is the absence of fit parameters: the only parameter entering the wave-function distribution is the total number N of propagating modes in the waveguides between the billiard and the outside world, which can be measured independently.

Single-wave-function statistics in open chaotic billiards, but without phase rigidity fluctuations, was first considered theoretically by Pnini and Shapiro [4] and subsequently by Ishio and co-workers [20,21]. Experimentally, wave functions in open billiards were investigated by Barth and Stöckmann [5] and by Kim *et al.* [6].

In Sec. II, we describe the calculation of the phase-rigidity distribution for a chaotic billiard. In Sec. III, we then use the phase-rigidity distribution to find wave-function distributions and correlations. In Sec. IV, we show that there are statistical correlations between wave functions that correspond to different scattering states. Such correlations are the open-cavity counterpart of correlations between different electronic wave functions in a weak magnetic field [19]. We conclude in Sec. V.

II. PHASE-RIGIDITY DISTRIBUTION

The key to the calculation of $P(\rho)$ in an open cavity is a relation between the scalar products of the in-cavity parts of scattering states ψ_μ and ψ_ν and the Wigner-Smith time-delay matrix Q [22],

$$\int_{\text{cavity}} d\mathbf{r}\psi_\mu(\mathbf{r})\psi_\nu^*(\mathbf{r}) = Q_{\mu\nu}, \tag{3}$$

where the scattering states have been normalized to unit incoming flux. Here the index $\mu=1,\ldots,N$ labels the waveguide and the transverse mode from which the field is injected into the cavity. The time-delay matrix $Q = -iS^\dagger \delta S/\delta\omega$ is the derivative of the scattering matrix S [23]. In order to calculate the scalar product $\rho_{\mu\mu}$ of the scattering state ψ_μ and its time-reversed ψ_μ^*, we perform a unitary transformation U that diagonalizes the Wigner-Smith time-delay matrix Q and rotates the scattering matrix S to the unit matrix [24],

$$S = U^{\mathrm{T}}U, \quad Q = U^\dagger \mathrm{diag}(\tau_1,\ldots,\tau_N)U. \tag{4}$$

The positive numbers τ_i, $i=1,\ldots,N$, are the "proper delay times," the eigenvalues of the Wigner-Smith time-delay matrix. If seen as a basis change, the transformation corresponding to the unitary matrix U both diagonalizes the scattering matrix and absorbs the scattering phases into the definition of the scattering states. Note that the incoming modes are transformed according to the unitary transformation U, while the outgoing modes transform according to U^*, as required by time-reversal symmetry.

In the transformed basis, the scattering matrix is diagonal and all scattering phase shifts are zero. Hence, in the transformed basis, the scattering states are standing waves, for which $\rho=1$. Transforming back to the original basis, we find

$$\rho_{\mu\mu} = \frac{\sum_{j=1}^{N} U_{j\mu}^2 \tau_j}{\sum_{j=1}^{N} |U_{j\mu}|^2 \tau_j}. \tag{5}$$

Let us now consider the statistical distribution of $\rho_{\mu\mu}$ for a chaotic billiard. Following previous works on this subject, we consider the parameter regime in which the frequency average is taken over a window $\Delta\omega \ll c/L \ll \omega$, where c is the velocity of wave propagation and L the size of the billiard, and in which the openings occupy only a small fraction of the billiard's boundary. It is only in this regime that wave functions have a universal distribution. We further limit ourselves to (quasi) two-dimensional billiards, in which the electric field perpendicular to the billiard plane is identified with the wave function ψ and the Poynting vector with the current density $\mathbf{j} \propto \mathrm{Im}\, \psi^* \boldsymbol{\nabla} \psi$ [25]. With these conditions, the joint distribution of the scattering matrix S and the Wigner-Smith time-delay matrix Q of a chaotic billiard is known from random-matrix theory [26]. The distribution of the proper time delays τ_i is [24,27]

$$P(\tau_1,\ldots,\tau_N) = \prod_{j=1}^{N} \theta(\tau_j)\tau_j^{-3N/2-1} e^{-N\tau_{\rm av}/2\tau_j} \prod_{i<j} |\tau_i - \tau_j|, \tag{6}$$

where $\tau_{\rm av}$ is the average delay time and $\theta(x)=1$ for $x>0$ and 0 otherwise, whereas the unitary matrix U is uniformly distributed in the group of unitary $N\times N$ matrices. Together with Eq. (5) this fixes the probability distribution $P(\rho_{\mu\mu})$. A direct consequence of Eqs. (5) and (6) and the uniform distribution of U in the group of unitary $N\times N$ matrices is that, for a chaotic cavity, $P(\rho_{\mu\mu})$ depends on the total number of propagating modes N summed over all waveguides only; it does not depend on how many waveguides are attached to the cavity or on how the total number of modes are distributed over the different waveguides. For example, the probability distribution $P(\rho_{\mu\mu})$ for a cavity with two double-mode waveguides is the same as that for a cavity attached to one single-mode waveguide and one triple-mode waveguide or a cavity with four single-mode waveguides.

We were able to obtain simple expressions for $P(\rho_{\mu\mu})$ in the limiting cases $N=2$ and $N\gg 1$. (The case $N=1$ is not relevant since there are no traveling waves for a billiard with one single-mode waveguide.) The distribution for $N=2$ is obtained parametrizing

$$U_{1\mu} = (1-T)^{1/2} e^{i\phi_1}, \quad U_{2\mu} = T^{1/2} e^{i\phi_2},$$

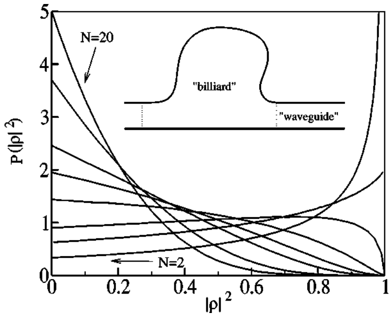

FIG. 1. Probability distribution of the phase rigidity $|\rho|^2$ for a wave function in an open chaotic billiard, for different numbers of propagating modes connecting the billiard to the outside world. From bottom to top at the left end of the figure, curves correspond to $N=2$, 3, 4, 6, 8, 10, 15, and 20. Inset: schematic drawing of billiard and waveguides.

where $0\leqslant T\leqslant 1$ and $0\leqslant\phi_{1,2}<2\pi$. A uniform distribution of the 2×2 unitary matrix U corresponds to a uniform distribution of T in the interval $0\leqslant T\leqslant 1$ and uniform distributions of the phases ϕ_1 and ϕ_2 [26]. Integrating over T, ϕ_1, ϕ_2, τ_1, and τ_2, we then find

$$P(\rho)=\frac{6+2(1-|\rho|^2)^{-1/2}}{3\pi(1+(1-|\rho|^2)^{1/2})^3},\quad 0\leqslant|\rho|<1. \tag{7}$$

Note that, although $P(\rho)$ is defined as a function of the complex variable ρ, $P(\rho)$ depends on the modulus $|\rho|$ only, as required by time translation invariance. For $N>2$ no such simple result could be obtained. A numerical evaluation of the probability distribution of the phase rigidity $|\rho|^2$ is shown in Fig. 1 for several values of N. In the limit $N\gg 1$, $P(\rho)$ approaches a Gaussian,

$$P(\rho)=\frac{N}{4\pi}e^{-N|\rho|^2/4}. \tag{8}$$

This is the same functional form as the phase-rigidity distribution for a quantum dot in a large uniform magnetic field [13,17,18].

III. LONG-RANGE WAVE-FUNCTION CORRELATIONS

Following Refs. [4,14,21], the joint distributions of intensities and current densities away from the boundary of the cavity for one wave function ψ_μ can be calculated from Berry's ansatz that ψ_μ can be written as a random superposition of plane waves [7],

$$\psi_\mu(\mathbf{r})=\sum_{\mathbf{k}} a_\mu(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{9}$$

In Eq. (9), all wave vectors $\mathbf{k}$ have the same modulus, while the amplitudes $a_\mu(\mathbf{k})$ are random complex numbers. For a closed cavity, amplitudes of time-reversed plane waves are related, $a_\mu(\mathbf{k})=e^{2i\phi}a_\mu(-\mathbf{k})^*$, where ϕ does not depend on $\mathbf{k}$. For an open cavity, no such strict relation exists. However, some degree of correlation between $a_\mu(\mathbf{k})$ and $a_\mu(-\mathbf{k})$ must persist in order to ensure the correct value of the scalar product of ψ_μ and ψ_μ^*, cf. Eq. (2) [21],

$$\rho_{\mu\mu}=\frac{\sum_{\mathbf{k}} a_\mu(\mathbf{k})a_\nu(-\mathbf{k})}{\sum_{\mathbf{k}} |a_\mu(\mathbf{k})|^2}. \tag{10}$$

(In the absence of correlations between $a_\mu(\mathbf{k})$ and $a_\mu(-\mathbf{k})$ one would have $\rho_{\mu\mu}=0$.) Taking the amplitudes corresponding to wavevectors pointing in different directions from identical and independent distributions, we see that Eq. (10) implies a relation between the second moments of the amplitude distribution,

$$\langle a_\mu(\mathbf{k})a_\mu(-\mathbf{k})\rangle=\rho_{\mu\mu}\langle|a_\mu(\mathbf{k})|^2\rangle. \tag{11}$$

This, together with the normalization condition $\Sigma_{\mathbf{k}}\langle|a(\mathbf{k})|^2\rangle=1/A$, where A is the area of the billiard, the central limit theorem, and the probability distribution $P(\rho_{\mu\mu})$ we calculated in the preceding section provides sufficient information to determine the full distribution of the wave function ψ.

As an example, we consider the joint distribution of the normalized intensity $I(\mathbf{r})=|\psi(\mathbf{r})|^2A$ and the magnitude of the normalized current density $J=|\mathbf{j}(\mathbf{r}')|$, $\mathbf{j}=(A/k)\mathrm{Im}\,\psi^*\boldsymbol{\nabla}\psi$ at the positions $\mathbf{r}$ and $\mathbf{r}'$, where $k=\omega/c$. If the statistical ensemble is generated by variation of the position $\mathbf{r}$ only, the single-wave-function distribution factorizes into separate probability distributions for I and J that each depends parametrically on the phase rigidity $|\rho|^2$ [14,21],

$$P_\rho[I(\mathbf{r}),J(\mathbf{r}')]=\frac{8J}{(1-|\rho|^2)^{3/2}}K_0\left(\frac{2J\sqrt{2}}{\sqrt{1-|\rho|^2}}\right)I_0\left(\frac{I|\rho|}{1-|\rho|^2}\right)\times\exp\left(-\frac{I}{1-|\rho|^2}\right), \tag{12}$$

where I_0 and K_0 are Bessel functions. When both position and frequency are varied to obtain the ensemble average, a further average over ρ is required,

$$P(I(\mathbf{r}),J(\mathbf{r}'))=\int d\rho P(\rho)P_\rho(I(\mathbf{r}),J(\mathbf{r}')). \tag{13}$$

After such average, $P(I,J)$ no longer factorizes in general. [The probability distribution $P(I,J)$ factorizes only if $P(\rho)$ is a δ function, which is the case for a closed billiard or a fully open billiard ($N\to\infty$) only.] The degree of correlations arising from the fluctuations of ρ is measured through the correlator

$$\langle I(\mathbf{r})^2J(\mathbf{r}')^2\rangle_c=-\frac{1}{2}\mathrm{var}\,|\rho|^2, \tag{14}$$

where $\langle AB\rangle_c=\langle AB\rangle-\langle A\rangle\langle B\rangle$ denotes the connected average. (Since normalization implies that $\langle I(\mathbf{r})\rangle=1$ for each wave function, correlators involving the first power of I factorize.) For a billiard with two single-mode waveguides,

$$\mathrm{var}|\rho|^2=\frac{8}{9}[148\ln 2-128(\ln 2)^2-41]\approx 0.078,$$

cf. Eq. (7). Similarly, we find for the correlator of intensities

$$\langle I(\mathbf{r})^2 I(\mathbf{r}')^2\rangle_c=\mathrm{var}|\rho|^2, \tag{15}$$

plus additional terms that describe short-range correlations.

IV. DIFFERENT SCATTERING STATES

Thus far we have studied the distribution of a single scattering state in an open billiard. However, for a billiard that is coupled to the outside world via, in total, N propagating modes, there are N orthogonal scattering states at each frequency. In this section we address the joint probability distribution of wave functions corresponding to different (and orthogonal) scattering states.

This question can be studied using the framework of Ref. [19], which generalizes the above considerations to the problem of correlations between wave functions. As before, the starting point is Berry's ansatz (9), with a different set of amplitudes $a_\mu(\mathbf{k})$ for each scattering state ψ_μ, $\mu=1,\dots,N$. We continue to take amplitudes $a_\mu(\mathbf{k})$ from identical and independent distributions for different directions of $\mathbf{k}$, whereas we allow for correlations between amplitudes of time-reversed waves and between amplitudes of different scattering states. Such correlations are necessary, because the in-cavity parts of different scattering states and their time-reversed states are not orthogonal, see, e.g., Eq. (3). (Uncorrelated amplitudes for different scattering states μ and ν would imply that wave functions in the cavity and their time reversed are orthogonal if they correspond to different scattering states.) Hence, the second moments of the amplitudes $a_\mu(\mathbf{k})$ should be chosen such that

$$n_{\mu\nu}\equiv\frac{\sum_{\mathbf{k}} a_\mu(\mathbf{k})a_\nu(\mathbf{k})^*}{\left[\sum_{\mathbf{k}}|a_\mu(\mathbf{k})|^2\right]^{1/2}\left[\sum_{\mathbf{k}}|a_\nu(\mathbf{k})|^2\right]^{1/2}}=\frac{\int d\mathbf{r}\psi_\mu(\mathbf{r})^*\psi_\nu(\mathbf{r})}{\left[\int d\mathbf{r}|\psi_\mu(\mathbf{r})|^2\int d\mathbf{r}'|\psi_\nu(\mathbf{r}')|^2\right]^{1/2}}, \tag{16}$$

$$\rho_{\mu\nu}\equiv\frac{\sum_{\mathbf{k}} a_\mu(\mathbf{k})a_\nu(-\mathbf{k})}{\left[\sum_{\mathbf{k}}|a_\mu(\mathbf{k})|^2\right]^{1/2}\left[\sum_{\mathbf{k}}|a_\nu(\mathbf{k})|^2\right]^{1/2}}=\frac{\int d\mathbf{r}\psi_\mu(\mathbf{r})\psi_\nu(\mathbf{r})}{\left[\int d\mathbf{r}|\psi_\mu(\mathbf{r})|^2\int d\mathbf{r}'|\psi_\nu(\mathbf{r}')|^2\right]^{1/2}}, \tag{17}$$

where, as before, the integrals are taken over the billiard only and we have chosen the normalization such that $n_{\mu\mu}=1$. Equations (16) and (17) then impose the following relations for second moments of the amplitude distributions:

$$\langle a_\mu(\mathbf{k})a_\nu(\mathbf{k})^*\rangle=n_{\mu\nu}\langle|a_\mu(\mathbf{k})|^2\rangle, \tag{18}$$

$$\langle a_\mu(\mathbf{k})a_\nu(-\mathbf{k})\rangle=\rho_{\mu\nu}\langle|a_\mu(\mathbf{k})|^2\rangle. \tag{19}$$

Repeating the same arguments as those leading to Eq. (5), we find that $n_{\mu\nu}$ and $\rho_{\mu\nu}$ can be expressed in terms of eigenvectors and eigenvalues of the time-delay matrix,

$$n_{\mu\nu}=\frac{\sum_j U_{j\mu}^*U_{j\nu}\tau_j}{\left(\sum_j|U_{j\mu}|^2\tau_j\sum_i|U_{i\nu}|^2\tau_i\right)^{1/2}},$$

$$\rho_{\mu\nu}=\frac{\sum_j U_{j\mu}U_{j\nu}\tau_j}{\left(\sum_j|U_{j\mu}|^2\tau_j\sum_i|U_{i\nu}|^2\tau_i\right)^{1/2}}. \tag{20}$$

The full distribution of the complex numbers $n_{\mu\nu}$ and $\rho_{\mu\nu}$ then follows from the known distributions of the $N\times N$ unitary matrix U and the proper time delays τ_j, $j=1,\dots,N$, see Sec. II. A simple expression is obtained in the limit $N\gg 1$, when $n_{\mu\nu}$ and $\rho_{\mu\nu}$ acquire a Gaussian distribution, with zero mean and with variance given by

$$\langle n_{\mu\nu}n_{\sigma\tau}\rangle=\frac{1}{N}\delta_{\mu\tau}\delta_{\nu\sigma}\quad\text{if}\quad\mu\neq\nu,$$

$$\langle\rho_{\mu\nu}\rho_{\tau\sigma}^*\rangle=\frac{2}{N}(\delta_{\mu\tau}\delta_{\nu\sigma}+\delta_{\mu\sigma}\delta_{\nu\tau}),$$

$$\langle n_{\mu\nu}\rho_{\sigma\tau}\rangle=\langle\rho_{\mu\nu}\rho_{\tau\sigma}\rangle=0. \tag{21}$$

Short-range correlations between different scattering modes arise from the fact that $\rho_{\mu\nu}$ and $n_{\mu\nu}$ are nonzero for $\mu\neq\nu$. These correlations exist if statistics is taken as a function of position only and if the ensemble also involves a frequency average. For example, for the second moment of the intensity and current density distributions, we find from Eq. (9)

$$\langle I_\mu(\mathbf{r})I_\nu(\mathbf{r}')\rangle_c=(|n_{\mu\nu}|^2+|\rho_{\mu\nu}|^2)J_0(k|\mathbf{r}-\mathbf{r}'|)^2,$$

$$\langle j_{\mu,\alpha}(\mathbf{r})j_{\nu,\beta}(\mathbf{r}')\rangle=\frac{1}{4}\delta_{\alpha\beta}(|n_{\mu\nu}|^2+|\rho_{\mu\nu}|^2)J_0(k|\mathbf{r}-\mathbf{r}'|)^2, \tag{22}$$

where $\alpha,\beta=x,y$. For the case $N=2$ of a billiard with two single-mode waveguides, the relevant expectation values $\langle|\rho_{12}|^2\rangle$ and $\langle|n_{12}|^2\rangle$ can be obtained from Eq. (20). Here, one parametrizes the unitary matrix U as

$$U_{11}=(1-T)^{1/2}e^{i\phi_1},$$

$$U_{12}=T^{1/2}e^{i\phi_3},$$

$$U_{21}=T^{1/2}e^{i\phi_2},$$

$$U_{22}=-(1-T)^{1/2}e^{i\phi_2+i\phi_3-i\phi_1},$$

where $0\leq T\leq 1$ and $0\leq\phi_{1,2,3}<2\pi$ are uniformly distributed [26]. Upon integration over T, ϕ_1, ϕ_2, ϕ_3, τ_1, and τ_2 one finds

$$\langle|\rho_{12}|^2\rangle=\frac{1}{15}(64\ln 2-37)\approx 0.49,$$

$$\langle|n_{12}|^2\rangle=\frac{1}{15}(26-32\ln 2)\approx 0.25.$$

Long-range correlations between wave functions of different scattering states arise from the fluctuations of the "scalar products" $n_{\mu\nu}$ and $\rho_{\mu\nu}$. They exist only if the ensemble involves a frequency average. The lowest moment with long-range correlations is

$$\langle I_\mu(\mathbf{r})^2 I_\nu(\mathbf{r}')^2\rangle_c=-2\langle I_\mu(\mathbf{r})^2 J_\nu(\mathbf{r}')^2\rangle_c=\langle|\rho_{\mu\mu}|^2|\rho_{\nu\nu}|^2\rangle_c, \tag{23}$$

where $\langle|\rho_{\mu\mu}|^2|\rho_{\nu\nu}|^2\rangle_c=-\langle|\rho_{\mu\mu}|^2\rangle\langle|\rho_{\nu\nu}|^2\rangle$. With a calculation similar to that of the short-range correlations one finds for $N=2$

$$\langle|\rho_{11}|^2|\rho_{22}|^2\rangle_c=\frac{8}{315}[5792\ln 2-4480(\ln 2)^2-1861]$$

$$\approx 0.032.$$

V. CONCLUSION

In conclusion, we have calculated the statistical distribution of wave functions in an open chaotic billiard. For an open billiard, the wave-function distribution that is obtained by an average over both frequency and position is different from the one obtained by an average over position only. In the latter case, which was considered in previous theoretical [4,20,21] and experimental [5,6] works, the wave-function distribution contains the wave function's phase rigidity as a fit parameter. For the full ensemble average considered here (both position and frequency are varied), no fit parameters are needed; the phase rigidity is a random quantity with a known probability distribution.

The fluctuations of the phase rigidities are responsible for long-range correlations between intensities and current densities in case of the full ensemble average. Long-range wave-function correlations were predicted previously for the real-to-complex crossover for electronic wave functions in a weak magnetic field [18]. Experimental verification of these effects would address aspects of random wave functions that go beyond a description in terms of a random superposition of plane waves. However, for the magnetic field-driven crossover in the electronic context as well as for the open billiards considered here, the relative magnitude of the long-range wave-function correlations is small, of the order of 10 percent or less [18,19]. Presently, the accuracy of solid-state experiments of wave functions in semiconductor quantum dots is insufficient to resolve such an effect [28]. The numerical smallness of the effect of phase rigidity fluctuations could also explain why intensity distributions in closed cavities with broken time-reversal symmetry measured by Chung *et al.* could not distinguish between theories with and without phase-rigidity fluctuations [10]. Our finding that the long-range correlations also exist in open microwave billiards, together with the availability of very precise measurements of wave-function distributions for this system [5,6], opens a new avenue for experimental observation of long-range wave-function correlations in the crossover ensemble and makes possible a fit-parameter-free comparison of experiment and theory.

ACKNOWLEDGMENTS

We thank Karsten Flensberg for important discussions. This work was supported by NSF under Grant No. DMR 0086509, and by the Packard foundation.

[1] H.J. Stöckmann, *Quantum Chaos: An Introduction* (Cambridge University Press, Cambridge, 1999).
[2] P. So, S.M. Anlage, E. Ott, and R.N. Oerter, Phys. Rev. Lett. **74**, 2662 (1995).
[3] U. Stoffregen, J. Stein, H.-J. Stöckmann, M. Kuś, and F. Haake, Phys. Rev. Lett. **74**, 2666 (1995).
[4] R. Pnini and B. Shapiro, Phys. Rev. E **54**, 1032 (1996).
[5] M. Barth and H.-J. Stöckmann, Phys. Rev. E **65**, 066208 (2002).
[6] Y.-H. Kim, M. Barth, U. Kuhl, and H.-J. Stoeckmann, e-print cond-mat/0301411.
[7] M.V. Berry, J. Phys. A **10**, 2083 (1977).
[8] T. Guhr, A. Müller-Groeling, and H.A. Weidenmüller, Phys. Rep. **299**, 189 (1998).
[9] K.B. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, Cambridge, 1997).
[10] S.-H. Chung, A. Gokirmak, D.-H. Wu, J.S.A. Bridgewater, E. Ott, T.M. Antonsen, and S.M. Anlage, Phys. Rev. Lett. **85**, 2482 (2000).
[11] D.H. Wu, J.S.A. Bridgewater, A. Gokirmak, and S.M. Anlage, Phys. Rev. Lett. **81**, 2890 (1998).
[12] J.B. French, V.K.B. Kota, A. Pandey, and S. Tomsovic, Ann. Phys. (N.Y.) **181**, 198 (1988).
[13] S.A. van Langen, P.W. Brouwer, and C.W.J. Beenakker, Phys. Rev. E **55**, 1 (1997).
[14] K. Życzkowski and G. Lenz, Z. Phys. B: Condens. Matter **82**, 299 (1991).
[15] E. Kanzieper and V. Freilikher, Phys. Rev. B **54**, 8737 (1996).
[16] H.-J. Sommers and S. Iida, Phys. Rev. E **49**, 2513 (1994).
[17] V.I. Fal'ko and K.B. Efetov, Phys. Rev. B **50**, 11 267 (1994).
[18] V.I. Fal'ko and K.B. Efetov, Phys. Rev. Lett. **77**, 912 (1996).
[19] S. Adam, P.W. Brouwer, J.P. Sethna, and X. Waintal, Phys. Rev. B **66**, 165310 (2002).
[20] H. Ishio, A.I. Saichev, A.F. Sadreev, and K.F. Berggren, Phys.

Rev. E **64**, 056208 (2001).

[21] A.I. Saichev, H. Ishio, A.F. Sadreev, and K.F. Berggren, J. Phys. A **35**, 87 (2002).

[22] F.T. Smith, Phys. Rev. **118**, 349 (1960).

[23] Formally, the derivative $\delta/\delta\omega$ is taken inside the cavity only, see, e.g., A.I. Baz', Y.B. Zel'dovich, and A.M. Perelomov, *Scattering, Reactions and Decay in Nonrelativistic Quantum Mechanics* (Israel Program for Scientific Translations, Jerusalem, 1969). However, for ergodic scattering from a chaotic cavity, there is no difference between the derivative $\delta/\delta\omega$ and a full derivative to frequency.

[24] P.W. Brouwer, K.M. Frahm, and C.W.J. Beenakker, Phys. Rev. Lett. **78**, 4737 (1997).

[25] P. Šeba, U. Kuhl, M. Barth, and H.-J. Stöckmann, J. Phys. A **32**, 8225 (1999).

[26] C.W.J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997).

[27] P.W. Brouwer, K.M. Frahm, and C.W.J. Beenakker, Waves Random Media **9**, 91 (1999).

[28] L.P. Kouwenhoven, C.M. Marcus, P.L. McEuen, S. Tarucha, R.M. Westervelt, and N.S. Wingreen, in *Mesoscopic Electron Transport*, Vol. 345 of *NATO Advanced Studies Institute, Series E: Applied Sciences*, edited by L.L. Sohn, L.P. Kouwenhoven, and G. Schön (Kluwer, Dordrecht, 1997).